9.9 Representation of Functions by Power Series

- **Find a geometric power series that represents a function.**
- **Construct a power series using series operations.**

Geometric Power Series

In this section and the next, you will study several techniques for finding a power series that represents a function. Consider the function

$$f(x) = \frac{1}{1-x}.$$

The form of f closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

In other words, when $a = 1$ and $r = x$, a power series representation for $1/(1-x)$, centered at 0, is

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.\end{aligned}$$

Of course, this series represents $f(x) = 1/(1-x)$ only on the interval $(-1, 1)$, whereas f is defined for all $x \neq 1$, as shown in Figure 9.22. To represent f in another interval, you must develop a different series. For instance, to obtain the power series centered at -1, you could write

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{1/2}{1-[(x+1)/2]} = \frac{a}{1-r}$$

which implies that $a = \frac{1}{2}$ and $r = (x+1)/2$. So, for $|x+1| < 2$, you have

$$\begin{aligned}\frac{1}{1-x} &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n \\ &= \frac{1}{2}\left[1 + \frac{(x+1)}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \cdots\right], \quad |x+1| < 2\end{aligned}$$

which converges on the interval $(-3, 1)$.

JOSEPH FOURIER (1768–1830)

Some of the early work in representing functions by power series was done by the French mathematician Joseph Fourier. Fourier's work is important in the history of calculus, partly because it forced eighteenth-century mathematicians to question the then-prevailing narrow concept of a function. Both Cauchy and Dirichlet were motivated by Fourier's work with series, and in 1837 Dirichlet published the general definition of a function that is used today.

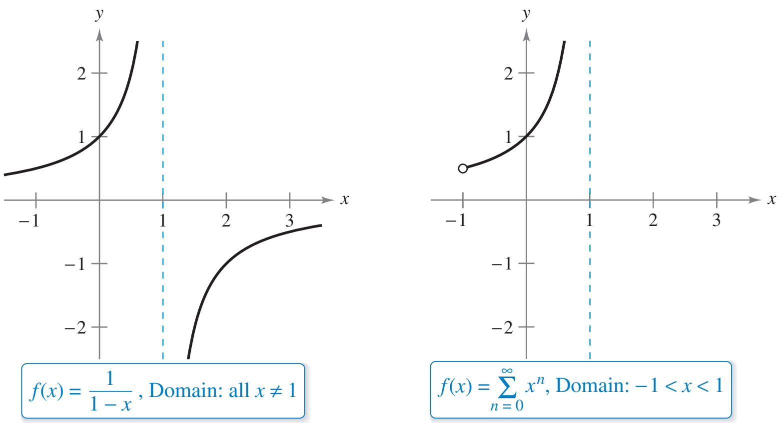

Figure 9.22

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EXAMPLE 1 Finding a Geometric Power Series Centered at 0

Find a power series for $f(x) = \dfrac{4}{x+2}$, centered at 0.

Solution Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{4}{2+x} = \frac{2}{1-(-x/2)} = \frac{a}{1-r}$$

which implies that $a = 2$ and

$$r = -\frac{x}{2}.$$

So, the power series for $f(x)$ is

$$\begin{aligned}\frac{4}{x+2} &= \sum_{n=0}^{\infty} ar^n\\ &= \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n\\ &= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).\end{aligned}$$

This power series converges when

$$\left|-\frac{x}{2}\right| < 1$$

which implies that the interval of convergence is $(-2, 2)$.

Long Division

$$\begin{array}{r} 2 - x + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \cdots \\ 2 + x\,\overline{)\,4 \qquad\qquad\qquad\qquad} \\ \underline{4 + 2x} \qquad\qquad\quad \\ -2x \qquad\qquad\quad \\ \underline{-2x - x^2} \qquad\quad \\ x^2 \qquad\quad \\ \underline{x^2 + \frac{1}{2}x^3} \quad \\ -\frac{1}{2}x^3 \quad \\ \underline{-\frac{1}{2}x^3 - \frac{1}{4}x^4} \end{array}$$

Another way to determine a power series for a rational function such as the one in Example 1 is to use long division. For instance, by dividing $2 + x$ into 4, you obtain the result shown at the left.

EXAMPLE 2 Finding a Geometric Power Series Centered at 1

Find a power series for $f(x) = \dfrac{1}{x}$, centered at 1.

Solution Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{a}{1-r}$$

which implies that $a = 1$ and $r = 1 - x = -(x - 1)$. So, the power series for $f(x)$ is

$$\begin{aligned}\frac{1}{x} &= \sum_{n=0}^{\infty} ar^n\\ &= \sum_{n=0}^{\infty} [-(x-1)]^n\\ &= \sum_{n=0}^{\infty} (-1)^n(x-1)^n\\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.\end{aligned}$$

This power series converges when

$$|x - 1| < 1$$

which implies that the interval of convergence is $(0, 2)$.

Operations with Power Series

The versatility of geometric power series will be shown later in this section, following a discussion of power series operations. These operations, used with differentiation and integration, provide a means of developing power series for a variety of elementary functions. (For simplicity, the operations are stated for a series centered at 0.)

Operations with Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$

2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$

3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

The operations described above can change the interval of convergence for the resulting series. For example, in the addition shown below, the interval of convergence for the sum is the *intersection* of the intervals of convergence of the two original series.

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{(-1,\,1)} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{(-2,\,2)} = \underbrace{\sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right) x^n}_{(-1,\,1)}$$

$$(-1, 1) \cap (-2, 2) = (-1, 1)$$

EXAMPLE 3 Adding Two Power Series

Find a power series for

$$f(x) = \frac{3x - 1}{x^2 - 1}$$

centered at 0.

Solution Using partial fractions, you can write $f(x)$ as

$$\frac{3x - 1}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x - 1}.$$

By adding the two geometric power series

$$\frac{2}{x + 1} = \frac{2}{1 - (-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x - 1} = \frac{-1}{1 - x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the power series shown below.

$$\frac{3x - 1}{x^2 - 1} = \sum_{n=0}^{\infty} [2(-1)^n - 1] x^n$$
$$= 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

The interval of convergence for this power series is $(-1, 1)$.

EXAMPLE 4 Finding a Power Series by Integration

Find a power series for

$$f(x) = \ln x$$

centered at 1.

Solution From Example 2, you know that

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n. \qquad \text{Interval of convergence: } (0, 2)$$

Integrating this series produces

$$\begin{aligned}\ln x &= \int \frac{1}{x}\,dx + C \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.\end{aligned}$$

By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\begin{aligned}\ln x &= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} \\ &= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.\end{aligned} \qquad \text{Interval of convergence: } (0, 2]$$

Note that the series converges at $x = 2$. This is consistent with the observation in the preceding section that integration of a power series may alter the convergence at the endpoints of the interval of convergence.

FOR FURTHER INFORMATION To read about finding a power series using integration by parts, see the article "Integration by Parts and Infinite Series" by Shelby J. Kilmer in *Mathematics Magazine.* To view this article, go to *MathArticles.com.*

In Section 9.7, Example 4, the fourth-degree Taylor polynomial for the natural logarithmic function

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

was used to approximate $\ln(1.1)$.

$$\begin{aligned}\ln(1.1) &\approx (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 \\ &\approx 0.0953083\end{aligned}$$

You now know from Example 4 in this section that this polynomial represents the first four terms of the power series for $\ln x$. Moreover, using the Alternating Series Remainder, you can determine that the error in this approximation is less than

$$\begin{aligned}|R_4| &\le |a_5| \\ &= \frac{1}{5}(0.1)^5 \\ &= 0.000002.\end{aligned}$$

During the seventeenth and eighteenth centuries, mathematical tables for logarithms and values of other transcendental functions were computed in this manner. Such numerical techniques are far from outdated, because it is precisely by such means that many modern calculating devices are programmed to evaluate transcendental functions.

EXAMPLE 5 Finding a Power Series by Integration

See LarsonCalculus.com for an interactive version of this type of example.

Find a power series for

$$g(x) = \arctan x$$

centered at 0.

Solution Because $D_x[\arctan x] = 1/(1 + x^2)$, you can use the series

$$f(x) = \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$ Interval of convergence: $(-1, 1)$

Substituting x^2 for x produces

$$f(x^2) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Finally, by integrating, you obtain

$$\begin{aligned}\arctan x &= \int \frac{1}{1 + x^2}\,dx + C \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.\end{aligned}$$

Let $x = 0$, then $C = 0$.

Interval of convergence: $(-1, 1)$

SRINIVASA RAMANUJAN (1887–1920)

Series that can be used to approximate π have interested mathematicians for the past 300 years. An amazing series for approximating $1/\pi$ was discovered by the Indian mathematician Srinivasa Ramanujan in 1914 (see Exercise 61). Each successive term of Ramanujan's series adds roughly eight more correct digits to the value of $1/\pi$. For more information about Ramanujan's work, see the article "Ramanujan and Pi" by Jonathan M. Borwein and Peter B. Borwein in *Scientific American*.

See LarsonCalculus.com to read more of this biography.

It can be shown that the power series developed for arctan x in Example 5 also converges (to arctan x) for $x = \pm 1$. For instance, when $x = 1$, you can write

$$\begin{aligned}\arctan 1 &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \\ &= \frac{\pi}{4}.\end{aligned}$$

However, this series (developed by James Gregory in 1671) does not give us a practical way of approximating π because it converges so slowly that hundreds of terms would have to be used to obtain reasonable accuracy. Example 6 shows how to use *two* different arctangent series to obtain a very good approximation of π using only a few terms. This approximation was developed by John Machin in 1706.

FOR FURTHER INFORMATION To read about other methods for approximating π, see the article "Two Methods for Approximating π" by Chien-Lih Hwang in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

EXAMPLE 6 Approximating π with a Series

Use the trigonometric identity

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

to approximate the number π [see Exercise 46(b)].

Solution By using only five terms from each of the series for $\arctan(1/5)$ and $\arctan(1/239)$, you obtain

$$4\left(4 \arctan \frac{1}{5} - \arctan \frac{1}{239}\right) \approx 3.1415926$$

which agrees with the exact value of π with an error of less than 0.0000001.

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9.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Geometric Power Series In Exercises 1–4, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.

1. $f(x) = \frac{1}{4 - x}$
2. $f(x) = \frac{1}{2 + x}$
3. $f(x) = \frac{4}{3 + x}$
4. $f(x) = \frac{2}{5 - x}$

Finding a Power Series In Exercises 5–16, find a power series for the function, centered at c, and determine the interval of convergence.

5. $f(x) = \frac{1}{3 - x}, \quad c = 1$
6. $f(x) = \frac{2}{6 - x}, \quad c = -2$
7. $f(x) = \frac{1}{1 - 3x}, \quad c = 0$
8. $h(x) = \frac{1}{1 - 5x}, \quad c = 0$
9. $g(x) = \frac{5}{2x - 3}, \quad c = -3$
10. $f(x) = \frac{3}{2x - 1}, \quad c = 2$
11. $f(x) = \frac{3}{3x + 4}, \quad c = 0$
12. $f(x) = \frac{4}{3x + 2}, \quad c = 3$
13. $g(x) = \frac{4x}{x^2 + 2x - 3}, \quad c = 0$
14. $g(x) = \frac{3x - 8}{3x^2 + 5x - 2}, \quad c = 0$
15. $f(x) = \frac{2}{1 - x^2}, \quad c = 0$
16. $f(x) = \frac{5}{5 + x^2}, \quad c = 0$

Using a Power Series In Exercises 17–26, use the power series

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

17. $h(x) = \frac{-2}{x^2 - 1} = \frac{1}{1 + x} + \frac{1}{1 - x}$
18. $h(x) = \frac{x}{x^2 - 1} = \frac{1}{2(1 + x)} - \frac{1}{2(1 - x)}$
19. $f(x) = -\frac{1}{(x + 1)^2} = \frac{d}{dx}\left[\frac{1}{x + 1}\right]$
20. $f(x) = \frac{2}{(x + 1)^3} = \frac{d^2}{dx^2}\left[\frac{1}{x + 1}\right]$
21. $f(x) = \ln(x + 1) = \int \frac{1}{x + 1}\,dx$
22. $f(x) = \ln(1 - x^2) = \int \frac{1}{1 + x}\,dx - \int \frac{1}{1 - x}\,dx$
23. $g(x) = \frac{1}{x^2 + 1}$
24. $f(x) = \ln(x^2 + 1)$
25. $h(x) = \frac{1}{4x^2 + 1}$
26. $f(x) = \arctan 2x$

Graphical and Numerical Analysis In Exercises 27 and 28, let

$$S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}.$$

Use a graphing utility to confirm the inequality graphically. Then complete the table to confirm the inequality numerically.

x	0.0	0.2	0.4	0.6	0.8	1.0
S_n						
$\ln(x + 1)$						
S_{n+1}						

27. $S_2 \le \ln(x + 1) \le S_3$
28. $S_4 \le \ln(x + 1) \le S_5$

Approximating a Sum In Exercises 29 and 30, (a) graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use 50 terms of the series to approximate the sum when $x = 0.5$, and (d) determine what the approximation represents and how good the approximation is.

29. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}$
30. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$

Approximating a Value In Exercises 31–34, use the series for $f(x) = \arctan x$ to approximate the value, using $R_N \le 0.001$.

31. $\arctan \frac{1}{4}$
32. $\int_0^{3/4} \arctan x^2\,dx$
33. $\int_0^{1/2} \frac{\arctan x^2}{x}\,dx$
34. $\int_0^{1/2} x^2 \arctan x\,dx$

Using a Power Series In Exercises 35–38, use the power series

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Find the series representation of the function and determine its interval of convergence.

35. $f(x) = \frac{1}{(1 - x)^2}$
36. $f(x) = \frac{x}{(1 - x)^2}$
37. $f(x) = \frac{1 + x}{(1 - x)^2}$
38. $f(x) = \frac{x(1 + x)}{(1 - x)^2}$

39. Probability A fair coin is tossed repeatedly. The probability that the first head occurs on the nth toss is $P(n) = \left(\frac{1}{2}\right)^n$. When this game is repeated many times, the average number of tosses required until the first head occurs is

$$E(n) = \sum_{n=1}^{\infty} nP(n).$$

(This value is called the *expected value of* n.) Use the results of Exercises 35–38 to find $E(n)$. Is the answer what you expected? Why or why not?

40. Finding the Sum of a Series Use the results of Exercises 35–38 to find the sum of each series.

(a) $\frac{1}{3}\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ (b) $\frac{1}{10}\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^n$

Writing In Exercises 41–44, explain how to use the geometric series

$$g(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

to find the series for the function. Do not find the series.

41. $f(x) = \dfrac{1}{1+x}$ **42.** $f(x) = \dfrac{1}{1-x^2}$

43. $f(x) = \dfrac{5}{1+x}$ **44.** $f(x) = \ln(1-x)$

45. Proof Prove that

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

for $xy \neq 1$ provided the value of the left side of the equation is between $-\pi/2$ and $\pi/2$.

46. Verifying an Identity Use the result of Exercise 45 to verify each identity.

(a) $\arctan \dfrac{120}{119} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

(b) $4 \arctan \dfrac{1}{5} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

[*Hint:* Use Exercise 45 twice to find $4 \arctan \frac{1}{5}$. Then use part (a).]

Approximating Pi In Exercises 47 and 48, (a) verify the given equation, and (b) use the equation and the series for the arctangent to approximate π to two-decimal-place accuracy.

47. $2 \arctan \dfrac{1}{2} - \arctan \dfrac{1}{7} = \dfrac{\pi}{4}$

48. $\arctan \dfrac{1}{2} + \arctan \dfrac{1}{3} = \dfrac{\pi}{4}$

Finding the Sum of a Series In Exercises 49–54, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

49. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n n}$ **50.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n n}$

51. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{5^n n}$ **52.** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n} \frac{1}{2n+1}$

53. $\displaystyle\sum_{n=0}^{\infty} (-1)^{n} \frac{1}{2^{2n+1}(2n+1)}$

54. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^{2n-1}(2n-1)}$

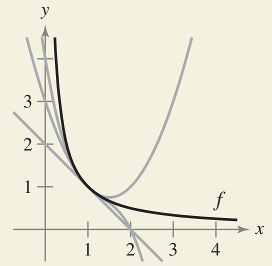
WRITING ABOUT CONCEPTS

55. Using Series One of the series in Exercises 49–54 converges to its sum at a much lower rate than the other five series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

56. Radius of Convergence The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is 3. What is the radius of convergence of the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$? Explain.

57. Convergence of a Power Series The power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x+1| < 4$. What can you conclude about the series $\sum_{n=0}^{\infty} a_n \dfrac{x^{n+1}}{n+1}$? Explain.

58. HOW DO YOU SEE IT? The graphs show first-, second-, and third-degree polynomial approximations P_1, P_2, and P_3 of a function f. Label the graphs of P_1, P_2, and P_3. To print an enlarged copy of the graph, go to *MathGraphs.com.*



Finding the Sum of a Series In Exercises 59 and 60, find the sum of the series.

59. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$ **60.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

61. Ramanujan and Pi Use a graphing utility to show that

$$\frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26{,}390n)}{(n!)396^{4n}} = \frac{1}{\pi}.$$

62. Find the Error Describe why the statement is incorrect.

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} \left(1 + \frac{1}{5}\right) x^n$$